

# TRANSCENDENTAL LIOUVILLE INEQUALITIES ON PROJECTIVE VARIETIES

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**ABSTRACT.** Let  $p$  be an algebraic point of a projective variety  $X$  defined over a number field. Liouville inequality tells us that the norm at  $p$  of a non vanishing integral global section of an hermitian line bundle over  $X$  is either zero or it cannot be too small with respect to the sup norm of the section itself. We study inequalities similar to Liouville's for subvarieties and for transcendental points of a projective variety defined over a number field. We prove that almost all transcendental points verify a good inequality of Liouville type. We also relate our methods to a (former) conjecture by Chudnovsky and give two applications to the growth of the number of rational points of bounded height on the image of an analytic map from a disk to a projective variety.

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## 1. INTRODUCTION

An important tool in diophantine geometry and in transcendental theory is the so called *Liouville inequality*. In its simplest form, it may be stated in the following way: Let  $n \in \mathbf{Z}$  and  $P(z) \in \mathbf{Z}[z]$  be a polynomial, then  $P(n) \neq 0$  implies that  $|P(n)| \geq 1$ . This can be generalized to any algebraic number and also to any algebraic point of a projective variety: Suppose that

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$\mathcal{X} \rightarrow \text{Spec}(\mathbf{Z})$  is a projective arithmetic scheme and  $\mathcal{L}$  is an hermitian ample line bundle on it. If  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ , we denote by  $\|s\|$  the supremum of  $\|s\|(z)$  on  $\mathcal{X}(\mathbf{C})$ . Then, Liouville inequality in this contest tells us that, if  $x \in \mathcal{X}(\overline{\mathbf{Q}})$  is an algebraic point, we can find a positive constant  $A$  such that, for every positive integer  $d$  and every  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  such that  $s(x) \neq 0$  we have  $\log \|s\|(x) \geq -A(\log \|s\| + d)$  cf. Theorem 3.1.

In section 3 we show that a similar inequality holds for sub varieties: Denote by  $X$  the generic fibre of the projective arithmetic scheme  $\mathcal{X}$  fixed above. For every closed sub variety  $Y$  of  $X(\mathbf{C})$  and every global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ , we denote by  $\|s\|_Y$  the supremum of  $\|s\|$  on  $Y$ . In section 3 we prove, cf. Theorem 3.2

**Theorem 1.1.** *Let  $Y$  be a Zariski closed sub variety of  $X(\mathbf{C})$  which is defined over  $\overline{\mathbf{Q}}$ , then we can find a constant  $A$  such that, for every positive integer  $d$  and every  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  which do not vanish identically on  $Y$  we have*

$$\log \|s\|_Y \geq -A(\log \|s\| + d).$$

Usually Liouville inequality is used, together with a Siegel lemma, some form of Schwartz inequalities and Zero Lemmas. In many diophantine proofs, The Zero Lemmas are used to ensure that the section we are dealing with do not vanish on involved algebraic point. Siegel lemma and Schwartz inequalities provide upper bounds for the value of the norm of a section in the point and Liouville inequality provides a lower bound of the same. When we apply the Zero Lemma, usually the price to pay is the effectivity of the statement: for instance, in the actual proofs, the effectivity of Roth theorem is lost in this way. If one could replace some of the involved algebraic points by transcendental points, probably the use of Zero Lemmas would become obsolete and we could gain in effectivity. Unfortunately this cannot be done directly because when one deals with transcendental points, Liouville inequality is not available anymore.

In this paper we will study some weak form of Liouville inequality which holds for "many" points of an algebraic variety.

In section 5 we study inequalities similar to Liouville inequality which are similar to the inequalities proposed by Chudnovsky, cf [5].

If  $P(z) \in \mathbf{Z}[z_1, \dots, z_N]$  is a polynomial, we will denote by  $\|P\|$  the maximum of the absolute values of its coefficients, cf. Theorem 5.2 .

In the paper [5] the author conjectured that the set of points  $\zeta \in \mathbf{C}^n$  for which there is a constant  $A$  (depending on  $\zeta$ ) such that, for every  $P(z)$  of degree  $d$ , we have  $\log |P(\zeta)| \geq -A(\log \|P\| + d)^{n+1}$  is full in  $\mathbf{C}^n$ . This conjecture have been proved by Amoroso [1].

Here we prove an inequality which is independent but may be stronger in some situations:

**Theorem 1.2.** *Suppose that  $N \geq 2$  and  $a \geq N + 1$ . A point  $\zeta \in \mathbf{C}^N$  is said to be of type  $S_a$  if we can find positive constants  $A_1$   $A_2$  and  $A_3$ , depending on  $\zeta$ , such that, for every non zero polynomial  $P(z) \in \mathbf{Z}[z_1, \dots, z_N]$  of degree  $d$  we have*

$$(1.1) \quad \log |P(\zeta)| \geq -A_1 d^a \log \|P\| - A_2 d^2 \log(d) - A_3 d.$$

*Then the set of points of type  $S_a$  is full in  $\mathbf{C}^N$ .*

We would like to remark that the proof of the theorem above do not use special properties of the field  $\mathbf{C}$ , thus it holds on every complete algebraically closed field. For instance it holds over  $p$ -adic fields. Remark that the proof by Amoroso in [1] holds only over  $\mathbf{C}$ . On the other side, the proof of Theorem 9.1 needs the application of Theorem 8.5 due to Sadullaev [14] Thm. 2.2, which makes a heavy use of peculiar properties of complex analysis. We think that a  $p$ -adic version of [14] would be very important and interesting.

In the following sections will study some possible inequalities of Liouville type which hold for almost all the transcendental points on a projective variety.

We will say that a point  $z \in \mathcal{X}(\mathbf{C})$  is *arithmetically generic* if, for every positive integer  $d$ , every non zero section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  do not vanish at  $z$ . It is easy to verify that arithmetic generic points do not depend on the line bundle  $\mathcal{L}$  but only on the generic fibre of  $\mathcal{X}$  and the complex embedding.

First of all we show that inequalities as good as Liouville's are not possible for transcendental points (cf. Theorem 6.7):

**Theorem 1.3.** *We can find a constant  $A$  depending only on  $(\mathcal{X}, \mathcal{L})$  for which the following holds: let  $z \in X_{\sigma_0}(\mathbf{C})$  be an arithmetically generic point, then, for every  $d \in \mathbb{N}$  there exists an infinite sequence of sections  $s_n \in H^0(\mathcal{X}, \mathcal{L}^d)$  for which*

$$(1.2) \quad \log \|s_n\|_{\sigma_0}(z) \leq -Ad^{\dim(X_K)}(\log^+ \|s_n\| + d).$$

For this reason we introduce the notion of  $S_a$  points, these are points for which an inequality as the one in the theorem above is weakened:

We suppose that the generic fiber of  $\mathcal{X}$  is of dimension  $N$  and that  $z \in \mathcal{X}(\mathbf{C})$ .

**Definition 1.4.** *Let  $a \geq N$  be a real number. We will say that  $z$  is of type  $S_a$  (or that  $z \in S_a(\mathcal{X})$ ) if we can find positive constants  $A = A(z, \mathcal{L}, a)$  and  $B = B(z, \mathcal{L}, a)$  depending on  $z, \mathcal{L}$  and  $a$  such that, for every positive integer  $d$  and every non zero global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  we have that*

$$\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -Ad^a(\log^+ \|s\| + Bd).$$

*In this case we will say that  $z$  has transcendental height on  $X_K$  (with respect to  $\mathcal{L}$  and  $a$ ) less or equal then  $A$ . Moreover we will denote by  $S_a(X_K)$  the subset of  $X_{\sigma_0}(\mathbf{C})$  of points of type  $S_a$ .*

Cf. definition 7.1. The definition "transcendental height" is due to the fact that, in the Liouville Inequality, which is similar, the involved constant  $A$  is essentially any real number which is bigger then the height of the algebraic point.

We will then prove:

**Theorem 1.5.** *For every  $a \geq N + 1$ , the set of points of type  $S_a$  is full for the Lebesgue measure on  $\mathcal{X}(\mathbf{C})$  (the complementary is of zero Lebesgue measure).*

Cf. Theorem 9.1. Usually this kind of theorem are proved coupling the classical Borel–Cantelli lemma (cf. 2.3) with an estimate of the volume of the set of elements having norm which do not satisfy the inequality of definition 7.1. The computation of this volume is usually

quite delicate and the standard strategy is to relate it to the distance of the points to the zero set of the involved global section. This is the strategy used for instance by Lang in [11], by Amoroso in [1] and in many proofs of the book [4]. Here we use a strategy which is a bit different: By Fubini Theorem we can reduce the proof to a similar computation on compact Riemann surfaces. Over one dimensional disks we use an argument involving Lagrange interpolation which allows to estimate, given a *polynomial*  $P(z)$ , the area of the set of points  $z$  for which  $|P(z)| \leq \epsilon$ . Then we approximate analytic functions by polynomials. In order to obtain good approximation, an important criterion of algebraicity due to Sadullaev [14] is used.

One should observe that inequalities of Theorem 1.2 seems better then inequalities of Theorem 9.1. Never the less they are obtained essentially the same methods. This due to the fact that the exponent of the degree in the involved inequalities is related to the arithmetic ampleness of the involved line bundle. When we work with polynomials with standard height, the involved line bundle is a arithmetically nef but not an arithmetically big line bundle. The fact that we can obtain better inequalities in this contest reflects this difference.

In the last section we describe two application to the growth of rational points of bounded height in the image of an analytic map of a disk to a projective variety. These are in the spirit of the principle explained above: the presence of a "good" transcendental point can be a tool which replaces Zero Lemmas.

Suppose that  $X$  is a smooth projective variety of dimension  $N > 1$  defined over a number field  $K$ . As before we suppose that we fixed a model  $\mathcal{X}$  of it and an hermitian ample line bundle  $\mathcal{L}$  over  $\mathcal{X}$ .

Let  $\Delta$  be the unit disk in  $\mathbf{C}$  and  $f : \Delta \rightarrow X(\mathbf{C})$  be an analytic map with Zariski dense image. Fix  $r < 1$  positive. We are interested in studying behavior of the cardinality  $C_r(f, T)$  of the set

$$(1.3) \quad S_r(f, T) := \{z \in \Delta / |z| < r ; f(z) \in X(K) \text{ and } h_{\mathcal{L}}(f(z)) \leq T\}$$

when  $T$  goes to infinity.

There is a huge literature on this problem: the classical theorem by Bombieri and Pila [2] tells us that in general  $C_r(f, T) \ll \exp(\epsilon T)$  but there are many interesting and natural conditions, which, if satisfied, imply that the growth of  $C_f(r, T)$  is polynomial in  $T$ , cf. for instance [6] or [3] and [12].

The theorems we can prove are:

**Theorem 1.6.** *Let  $a \geq N + 1$ . Suppose that there is  $z_0 \in \Delta$  such that  $f(z_0) \in S_a(X)$ . Then, for every  $\epsilon > 0$  and  $\gamma > \frac{1}{n}$  there exists a constant  $A = A(\mathcal{X}, \mathcal{L}, r, f, \epsilon, \gamma, a)$  such that, if  $T \geq A$  then we have*

$$(1.4) \quad C_f(r, T) \leq \epsilon T^{\gamma(a+1)+1}.$$

and

**Theorem 1.7.** *Let  $f : \Delta_1 \rightarrow X(\mathbf{C})$  as before. Let  $a \geq n - 1$  be a real number. Let  $s_0 \in H^0(\mathcal{L}, \mathcal{L})$  be an irreducible smooth divisor. Suppose that there is  $p \in f(\Delta_r) \cap \text{div}(s_0)$  which is of type  $S_a(\text{div}(s_0))$ . Then, for every  $\epsilon > 0$  and  $\gamma > \frac{1}{n}$  there exists a constant  $A = A(\mathcal{X}, \mathcal{L}, r, f, \epsilon, \gamma)$  such that, if  $T \geq A$  then we have*

$$(1.5) \quad C_f(r, T) \leq \epsilon T^{\gamma(a+1)+1}.$$

These theorems should also be compared with a similar theorem proved by Surroca [15]. In her theorem there is no condition on the type of points contained in the image but the conclusion is that the polynomial growth holds only for a sequence of values of  $T$ . Her paper provides also counterexamples which show that her theorem is sharp. Her counterexamples provide then examples of maps which do not verify the hypothesis of Theorems 1.6 and 1.7. The reader should also compare with the paper [3] where similar ideas are exploited.

The proof of Theorems 1.6 and 1.7 exploit the generalized Liouville inequality for points of type  $S_a$  plus a Siegel Lemma which bounds the degree  $d$  and the height of a section of  $\mathcal{L}^d$  vanishing on  $S_r(f, T)$ : cf. 10.2.

## 2. NOTATIONS AND BASIC FACTS ON ARAKELOV GEOMETRY

**2.1. Tools and notations from arithmetic geometry and Arakelov theory.** Let  $K$  be a number field and  $O_K$  be its ring of integers. We will denote by  $M_K^\infty$  the set of infinite places of  $K$ . We fix a place  $\sigma_0 \in M_K^\infty$ .

Let  $X_K$  be a projective variety of dimension  $n$  defined over  $K$ .

If  $\tau \in M_K^\infty$  and  $F$  is an object over  $X_K$  ( $F$  may be a sheaf, a divisor, a cycle...), we will denote by  $X_\tau$  the complex variety  $X_K \otimes_\tau \mathbf{C}$  and by  $F_\tau$  the restriction of  $F$  to  $X_\tau$ .

A model  $\mathcal{X} \rightarrow \text{Spec}(O_K)$  of  $X_K$  is a flat projective  $O_K$  scheme whose generic fiber is isomorphic to  $X_K$ . Suppose that  $L_K$  and  $\mathcal{X}$  are respectively a line bundle over  $X_K$  and a model of it; We will say that a line bundle  $\mathcal{L}$  over  $\mathcal{X}$  is a model of  $L_K$  if its restriction to the generic fiber is isomorphic to  $L_K$ .

If  $\mathcal{X}$  is a model of  $X_K$ , an hermitian line bundle  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_\sigma)_{\sigma \in M_K^\infty}$  is a line bundle over it equipped, for every  $\tau \in M_K^\infty$  a metric on  $L_\tau$  with the condition that, if  $\sigma = \overline{\tau}$  then the metric over  $L_\sigma$  is the conjugate of the metric on  $L_\tau$ .

If  $X_K$  is a projective variety, it is easy to see that for every line bundle  $L_K$  on  $X_K$ , we can find an embedding  $\iota : X_K \hookrightarrow P_K$ , where  $P_K$  is a smooth projective variety and  $L = \iota^*(M)$  with  $M$  line bundle on  $P_K$ . A metric on  $L$  will be said to be smooth if it is the restriction of a smooth metric on  $M$ .

Let  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_\sigma)_{\sigma \in M_K^\infty}$  be an hermitian line bundle on a model  $\mathcal{X}$  of  $X_K$ . If  $s \in H^0(X_K, L_K^d)$  is a non zero section, we will denote by  $\log^+ \|s\|$  the real number  $\sup_{\tau \in M_K^\infty} \{0, \log \|s_\tau\|_\tau\}$ . More generally, if  $a$  is a real number, we will denote by  $a^+$  the real number  $\sup\{a, 0\}$  and by  $a_+$  the real number  $\sup\{1, a\}$ .

**Definition 2.1.** *An arithmetic polarization  $(\mathcal{X}, \overline{\mathcal{L}})$  of  $X_K$  is the choice of the following data:*

- An ample line bundle  $L_K$  over  $X_K$
- A projective model  $\mathcal{X} \rightarrow \text{Spec}(O_K)$  of  $X_K$  over  $O_K$ .

- A relatively ample line bundle hermitian line bundle  $\overline{\mathcal{L}}$  over  $\mathcal{X}$  which is a model of  $L_K$ .
- For every  $\tau \in M_K^\infty$  we suppose that the metric on  $L_\tau$  is smooth and positive.

We recall the following standard facts of Arakelov theory:

- If  $L$  is an hermitian line bundle over  $\text{Spec}(O_K)$  and  $s \in L$  is a non vanishing section, we define

$$(2.1) \quad \widehat{\deg}(L) := \log(\text{Card}(L/sO_K)) - \sum_{\sigma \in M_K^\infty} \log \|s\|.$$

If  $E$  is an hermitian vector bundle of rank  $r$  on  $\text{Spec}(O_K)$ , we define  $\widehat{\deg}(E) := \widehat{\deg}(\wedge^r E)$  and the slope of  $E$  is  $\widehat{\mu}(E) = \frac{\widehat{\deg}(E)}{r}$ .

- Within all the sub bundles of  $E$  there is one whose slope is maximal, we denote by  $\widehat{\mu}_{\max}(E)$  its slope. It is easy to verify that  $\widehat{\mu}_{\max}(E_1 \oplus E_2) = \max\{\widehat{\mu}_{\max}(E_1), \widehat{\mu}_{\max}(E_2)\}$ .
- We will need the following version of the Siegel Lemma:

**Lemma 2.2.** (Siegel Lemma) Let  $E_1$  and  $E_2$  be hermitian vector bundles over  $O_K$ . Let  $f : E \rightarrow E_2$  be a non injective linear map. Denote by  $m = \text{rk}(E_1)$  and  $n = \text{rk}(\text{Ker}(f))$ . Suppose that there exists a positive real constant  $C$  such that:

- $E_1$  is generated by elements of sup norm less or equal then  $C$ .
- For every infinite place  $\sigma$  we have  $\|f\|_\sigma \leq C$

Then there exists a non zero element  $v \in \text{Ker}(f)$  such that

$$(2.2) \quad \sup_{\sigma \in M_K^\infty} \{\log \|v\|_\sigma\} \leq \frac{m}{n} \log(C^2) + \left(\frac{m}{n} - 1\right) \widehat{\mu}_{\max}(E_2) + 3 \log(n) + A$$

where  $A$  is a constant depending only on  $K$ .

A proof of this version of Siegel Lemma can be found in [7].

- Let  $L$  be an hermitian ample line bundle on a projective variety  $Z$  equipped with a smooth metric  $\omega$ . We suppose that the metric on  $L$  is smooth. Over  $H^0(Z, L^d)$  we can define two natural norms:

$$(2.3) \quad \|s\|_{\text{sup}} := \sup_{z \in Z} \|s\|(z) \quad \text{and} \quad \|s\|_{L^2} := \sqrt{\int_Z \|s\|^2 \omega^n}.$$

These norms are comparable: we can find constants  $C_i$  such that

$$(2.4) \quad C_1 \|s\|_{L^2} \leq \|s\|_{\text{sup}} \leq C_2^d \|s\|_{L^2}.$$

This statement (due to Gromov) is proved for instance in [16] Lemma 2 p. 166 when  $Z$  is smooth. The general statement can be deduced by taking a resolution of singularities (remark that the proof of [16] Lemma 2 p. 166 do not require that  $L$  is ample).

- Suppose that  $L$  is an hermitian ample line bundle on a smooth projective variety  $X$  defined over  $\mathbf{C}$ . Then we can find a constant  $C$  for which the following holds: for every couple of positive integers  $d_1$  and  $d_2$  and non vanishing global sections  $s_1 \in H^0(X, L^{d_1})$  and  $s_2 \in H^0(X, L^{d_2})$  we have

$$(2.5) \quad \log \|s_1\|_{\sup} + \log \|s_2\|_{\sup} \geq \log \|s_1 \cdot s_2\|_{\sup} \geq (d_1 + d_2)C + \log \|s_1\|_{\sup} + \log \|s_2\|_{\sup}.$$

- If  $(\mathcal{X}, \mathcal{L})$  is an arithmetic polarization of  $X_K$ , then we can find constants  $C_1$  and  $C_2$  such that

$$(2.6) \quad C_1^{d^{n+1}} T^{d^n} \leq \text{Card} \left( \left\{ s \in H^0(\mathcal{X}, \mathcal{L}^d) \mid \sup_{\tau \in M_K^\infty} \{\|s\|_\tau\} \leq T \right\} \right) \leq C_2^{d^{n+1}} T^{d^n}.$$

This is a consequence of [17], Theorem 1.4, [8] Theorem 2 and the comparison above.

- If  $\mathcal{L}$  is an arithmetically ample line bundle, then for  $d$  sufficiently big, the lattice  $H^0(\mathcal{X}, \mathcal{L})$  is generated by sections of sup norm less or equal then one. Cf. [17] for a proof.

**2.2. Tools and notations from measure theory.** If  $X$  is a variety and  $A \subset X$  is a subset. We will say that  $A$  is *full in  $X$*  if the Lebesgue measure of  $X \setminus A$  is zero.

A key tool in this paper is the classical Theorem of Borel–Cantelli, which can be found in any standard book in measure theory:

**Proposition 2.3.** *Let  $X$  be a variety equipped with the Lebesgue measure  $\mu$ . Let  $\{A_n\}_{n \in \mathbb{N}}$  be a sequence of measurable sets of  $X$  such that*

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

then

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k\right) = 0.$$

That means that almost all  $x \in X$  belong only to finitely many  $A_n$ .

### 3. LIOUVILLE LOWER BOUND FOR SUB VARIETIES

We fix a complex embedding  $\sigma_0 \in M_K$ . Suppose that  $X_K$  is a projective variety and  $(\mathcal{X}, \mathcal{L})$  is an arithmetic polarization over it. The very definition of the height of an algebraic point gives us a lower bound of the value of a global section of  $\mathcal{L}^d$  on it in terms of the its sup norm:

**Theorem 3.1.** *Let  $p \in X_K(\overline{K})$  be an algebraic point. Let  $p_0 \in X_{\sigma_0}$  be its image. Then we can find a constant  $A$ , depending on  $X_K$ ,  $p$  and the arithmetic polarization for which the following holds: for every positive integer  $d$  and global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  such that  $s(p) \neq 0$  we have*

$$(3.1) \quad \log \|s\|_{\sigma_0}(p_0) \geq -A(\log^+ \|s\| + d).$$

*Proof.* Denote by  $G$  the Galois group of  $\overline{K}$  over  $K$ . Let  $N = [K(p) : K]$  and  $\{p = p_0, p_1, \dots, p_N\}$  the orbit of  $p$  under  $G$ . For every  $\sigma \in M_K^\infty$  denote by  $p_i^\sigma$  the image of  $p_i$  in  $X_\sigma$ .

Let  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  not vanishing at  $p$ . By definition of height (computed as Arakelov degree), we have that

$$(3.2) \quad dh_L(p) \geq - \sum_{\sigma \in M_K^\infty} \sum_{i=1}^N \log \|s\|_\sigma(p_i^\sigma).$$

From this we obtain

$$(3.3) \quad \log \|s\|_{\sigma_0}(p_0) \geq -dh_L(p) - \sum_{i=1}^N \log \|s\|_{\sigma_0}(p_i^{\sigma_0}) - \sum_{\tau \neq \sigma_0} \sum_{i=0}^N \log \|s\|_\tau(p_i^\tau).$$

The conclusion easily follows.  $\square$

We would now like to show that a similar inequality holds for closed sub varieties of  $X_K$ : If  $Z \subset X_K$  is a closed sub variety and if  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  is a non zero section non vanishing identically on  $Z$ , we will denote

$$\log \|s\|_{Z, \sigma_0} := \sup\{\log \|s\|_{\sigma_0}(z) / z \in Z_{\sigma_0}(\mathbf{C})\}.$$

**Theorem 3.2.** *Let  $Y_K$  be a closed sub variety of  $X_K$ , then we can find a constant  $A$ , depending on  $Y_K$ ,  $X_K$  and the arithmetic polarization for which the following holds: For every positive integer  $d$  and global section  $s \in H^0(\mathcal{X}, cL^d)$  not vanishing on  $Y_K$*

$$(3.4) \quad \log \|s\|_{Y_K, \sigma_0} \geq -A(\log^+ \|s\| + d).$$

*Proof.* In order to prove the theorem we need the following lemma:

**Lemma 3.3.** *Let  $Z$  be a projective variety and  $L$  an ample line bundle over it. Let  $h_L(\cdot)$  be a height associated to  $L$ . Then there exists a Zariski dense set of points  $S \subset Z(\overline{K})$  and a constant  $C$  such that, for every  $p \in S$  we have*

$$(3.5) \quad \frac{h_L(p)}{[K(p) : K]} \leq C.$$

Let's show first how the lemma implies the theorem: Let  $s$  be the global section as in the Theorem. By the Lemma, we can find a point  $p \in S$  such that  $s(p) \neq 0$ . Now the proof follows the proof of Theorem 3.1: using the notation of Theorem 3.1, we have that

$$(3.6) \quad dh_L(p) \geq - \sum_{\sigma \in M_K^\infty} \sum_{i=1}^N \log \|s\|_\sigma(p_i^\sigma).$$

Thus

$$\begin{aligned} [K(p) : K] \log \|s\|_{Y, \sigma_0} &\geq -dh_L(P) - [K(p) : \mathbf{Q}] \log^+ \|s\| \\ &\geq -C[K(p) : K] - [K(p) : K][K : \mathbf{Q}] \log^+ \|s\|. \end{aligned}$$



The conclusion follows.  $\square$

*Proof.* (of Lemma 3.3) we first deal with the case when  $X_K = \mathbf{P}^N$ ,  $L = \mathcal{O}(1)$  and the height is the standard Weil Height. In this case, it suffices to observe that the set of points having homogeneous coordinates which are roots of unity will satisfy the conclusion of the theorem. Denote by  $S_{\mathbf{P}}$  this set.

In order to attack the general case, it suffices to observe that the set  $S$  (if it exists) is independent on the choice of the ample line bundle and the representative of the height. Thus, let  $p : X_K \rightarrow \mathbf{P}^{\dim(X_K)}$  be a finite morphism and  $L := p^*(\mathcal{O}(1))$ . The line bundle  $L$  is ample and  $S := p^{-1}(S_{\mathbf{P}})$  will satisfy the conclusion of the Lemma.  $\square$

Theorems 3.1 and 3.2 tell us that the algebraic sub varieties defined over the algebraic closure of  $K$  satisfy what we would call a *Liouville property*: non vanishing integral sections cannot be too small over them. This is evidently similar to the classical Liouville property of algebraic numbers which is one of the main tools in diophantine approximation and transcendence theory. On the other side, one can find sub varieties which are not defined over  $\overline{K}$  and for which a similar property do not hold. For instance there exist points  $\alpha$  of  $\mathbb{P}^1(\mathbf{C})$  for which we can find a sequence of polynomials  $P_n(z) \in \mathbf{Z}[z]$  and a diverging sequence  $\omega_n$  for which  $\log \|P_n(\alpha)\| \leq -\omega_n \log \|P_n\|$ .

In the following sections we will study what kind of inequalities of "Liouville type" we can hope on transcendental points.

#### 4. AN AREA COMPUTATION ON DISKS

In this section we will prove a technical Lemma which will be used in the paper. We want to compute the area of the points of a disk where an analytic function is "small".

In the sequel, we will denote by  $\Delta_r$  the disk  $\{z \in \mathbf{C} / |z| < r\}$ . An hermitian line bundle  $L$  on  $\Delta_r$  is just the trivial line bundle  $\mathcal{O}_{\Delta_r}$  equipped with a non vanishing smooth positive function  $\rho(z) := \|1\|(z)$ . If  $f \in H^0(\Delta_r, L^d)$  then  $\|f\|(z) = |f|(z) \cdot \rho^d(z)$ . If  $L$  is an hermitian line bundle on  $\Delta_r$ , for every  $r_0 < r$  and section  $f \in H^0(\Delta_r; L)$ , we will denote  $\|f\|_{r_0} := \sup\{\|f\|(z) / z \in \Delta_{r_0}\}$  (if  $\rho = 1$  then we write  $\|f\|_r = |f|_r$ ).

Let  $C > 1$  be a fixed real number. and  $r_0 < r_1 < r$ . For every positive integer  $d$ , let

$$(4.1) \quad B_C(r_0, r_1, L^d) := \{f \in H^0(\Delta_r, L^d) / \|f\|_{r_1} \leq C^d \|f\|_{r_0}\}.$$

Observe that  $B_C(r_0, r_1, L^d)$  is different from zero only if  $d \geq 0$  and that, a priori, it is just a set (it may be not closed for the sum).

Since every line bundle on  $\Delta$  is trivial, if  $M$  is another hermitian line bundle on  $\Delta$  and  $\varphi : L \rightarrow M$  is an isomorphism (not necessarily hermitian), then there is a constant  $A$  such that, for every  $d$ ,

$$(4.2) \quad \varphi^d(B_C(r_0, r_1, L^d)) = B_A(r_0, r_1, M^d).$$

Consequently, the set  $B_C(r_0, r_1, L^d)$  is essentially independent on the chosen metric on  $L$ . A classical Theorem by Bernstein tells us that it contains the ring of polynomials,

the Theorem by Sadullaev [14] which will be used in section 8, essentially characterize it.

Let  $\mu(\cdot)$  be the Lebesgue measure on  $\Delta_r$ . For every positive integer  $\epsilon$  and global section  $f \in H^0(\Delta_1, L^d)$  denote

$$V(f, r_0, \epsilon) := \mu(\{z \in \Delta_{r_0} / \|f\|(z) < \epsilon \|f\|_{r_0}\}).$$

The first lemma is a sort of reduction to the case of polynomials.

**Lemma 4.1.** *Fix two positive numbers  $r_0 < r_1 < 1$ . Let  $L$  be an hermitian line bundle on  $\Delta_1$ . Fix a positive constant  $C > 1$ . We can find a positive constant  $C_1 = C_1(L, r_0, r_1, C)$  depending only on  $L$ ,  $r_0$ ,  $r_1$  and  $C$  with the following property: for every  $f \in B_C((r_0, r_1, L^d)$ , every positive integer  $\epsilon$  and every sufficiently big positive integer  $\beta$  we have*

$$V(f, r_0, \epsilon) \leq C_1 \beta d \epsilon^{2/\beta d}.$$

*Proof.* We first recall the following standard fact about interpolation of holomorphic functions on disks, the proof of which can be found in any standard book in complex analysis:

**Fact 4.2.** *We can find positive constants  $C_0$  and  $\alpha < 1$  depending only on  $r_0$  and  $r_1$  with the following property: Let  $f \in \mathcal{O}_{\Delta_r}$  and  $x_0, \dots, x_N$  be  $N+1$  points in  $\Delta_{r_0}$ . Then there exists a polynomial  $P(z)$  for which the following holds:*

- a)  $\deg(P(z)) \leq N$ ;
- b)  $P(x_i) = f(x_i)$  for every  $i = 0, \dots, N$ ;
- c)  $|f(z) - P(z)|_{r_0} \leq C_0 \alpha^{N+1} |f|_{r_1}$ .

We fix a sufficiently big positive integer  $\beta$  (we will determine how big it should be at the end of the proof). Let  $f \in B_C((r_0, r_1, L^d)$ . We will denote by  $V$  the real number  $V(f, r_0, \epsilon)$  and by  $A$  the set  $\{z \in \Delta_{r_0} / \|f\|(z) < \epsilon \|f\|_{r_0}\}$ . We can find points  $x_0, \dots, x_{\beta d} \in \Delta_{r_0}$  such that:

- a)  $\|f\|(x_i) \leq \epsilon \|f\|_{r_0}$ ;
- b)  $|x_i - x_j| \geq \sqrt{\frac{V}{\pi(\beta d + 1)}}$  for  $i \neq j$ .

For each  $j = 0; \dots; \beta d$ , let  $Q_j(z) := \frac{\prod_{i \neq j} (z - x_i)}{\prod_{i \neq j} (x_i - x_j)}$ .

Let  $P(z)$  be the polynomial for which properties (a), (b) and (c) of Fact 4.2 with respect to the holomorphic function  $f$  and points  $x_0, \dots, x_{\beta d}$  holds.

We can find constants  $C_2$  and  $C_3$ , depending only on  $L$  and the  $r_i$ 's for which the following holds: See  $P(z)$  as a section of  $L^d$  then

$$\|f(z) - P(z)\|_{r_0} \leq C_0 \cdot C_2^d \cdot \alpha^{\beta d + 1} \|f\|_{r_1} \leq C_3^d \cdot \alpha^{\beta d + 1} \|f\|_{r_0}.$$

The last inequality follows from the fact that  $f \in B_C((r_0, r_1, L^d)$ .

Fix  $\beta$  such that  $C_3 \cdot \alpha^\beta \leq 1/2$ . With this choice we have that

$$\|f\|_{r_0} \leq 2 \|P(z)\|_{r_0}.$$

Since the set  $\{Q_j(z)\}$  is a basis of the space of polynomials of degree less or equal then  $\beta d$ , we can write

$$P(z) = \sum_{j=0}^{\beta d} f(x_j) Q_j(z).$$

Thus we can find a constant  $C_4$  such that

$$\|P(z)\|_{r_0} \leq (\beta d + 1) \cdot (\pi \cdot (\beta d + 1))^{(\beta d)/2} \cdot C_4^d \cdot \epsilon \cdot \|f\|_{r_0} \cdot \left( \frac{2r_0}{\sqrt{V}} \right)^{\beta d}.$$

Hence we can find a constant  $C_1$  such that

$$V \cdot \|f\|_{r_0}^{2/\beta d} \leq C_1 \beta d \epsilon^{2/\beta d} \|f\|_{r_0}^{2/\beta d}.$$

The conclusion follows.  $\square$

As a special case of the Lemma we find:

**Corollary 4.3.** *We can find a constant  $C$  for which the following holds: For every  $\epsilon > 0$  and polynomial  $P(z) \in \mathbf{C}[z]$  of degree  $d$ , denoting by  $\mu(\cdot)$  the Lebesgue measure in  $\mathbf{C}$ , we have*

$$(4.3) \quad \mu(\{z \in \Delta_r \mid |P(z)| \leq \epsilon \|P(z)\|_r\}) \leq C d \epsilon^{2/d}.$$

## 5. ABOUT A CONJECTURE BY CHUDNOVSKY

In this section we will prove an inequality in the direction (former) conjecture by Chudnovsky.

If  $P(z) \in \mathbf{C}[z_1, \dots, z_N]$  is a polynomial, we will denote by  $\|P\|$  the maximum of the absolute values of its coefficients.

In the paper [1] the author proves that the set of points  $\zeta \in \mathbf{C}^N$  for which we can find a constant  $A$ , such that, for every non zero polynomial  $P(z) \in \mathbf{Z}[z_1, \dots, z_N]$  of degree  $d$ , we have  $\log |P(\zeta)| \geq -A(\log \|P\| + d)^{N+1}$ , is full in  $\mathbf{C}^N$ . This was conjectured by Chudnovsky in [5].

In this section we will prove a similar inequality, in the same contest. The inequality we prove is independent of the one proved by Amoroso. It is weaker in the degree of the polynomial but stronger in the height of them.

**Definition 5.1.** *a) A complex number  $\zeta \in \mathbf{C}$  is said to be of type  $S_1$  if we can find a positive constant  $A$  such that for every non zero polynomial  $P(z) \in \mathbf{Z}[z]$  of degree  $d$  we have*

$$(5.1) \quad \log |P(\zeta)| \geq -Ad(\log \|P\| + \log(d) + 1).$$

*a.1) In this case we will say that  $\zeta$  has transcendental Weil height less or equal then  $A$  on  $\mathbf{P}^1$ .*

b) Suppose that  $N \geq 2$  and  $a \geq N + 1$ . A point  $\zeta \in \mathbf{C}^N$  is said to be of type  $S_a$  if we can find positive constants  $A_1$ ,  $A_2$  and  $A_3$ , depending on  $\zeta$ , such that, for every non zero polynomial  $P(z) \in \mathbf{Z}[z_1, \dots, z_N]$  of degree  $d$  we have

$$(5.2) \quad \log |P(\zeta)| \geq -A_1 d^a \log \|P\| - A_2 d^2 \log(d) - A_3 d.$$

b.1) In this case we will say that  $\zeta$  has transcendental Weil height less or equal then  $A$  on  $\mathbf{P}^N$ .

Of course, the analogue of (b) for  $N = 1$  in the definition, is weaker then (a).

The main theorem of this section is the following:

**Theorem 5.2.** *For every  $a \geq N + 1$ , the set of points of type  $S_a$  is full in  $\mathbf{C}^N$ .*

Before we start the proof we recall the following fact:

– Denote by  $V_d(N)$  the lattice of polynomial of degree at most  $d$  in  $N$  variables with integer coefficients. Then we can find constants  $B_i$  (depending only on  $N$ , but independent on  $d$ ) such that, for every positive constant  $H$  we have:

$$(5.3) \quad \text{Card}(\{P(z) \in V_d(N) \mid \|P(z)\| \leq H\}) \leq B_1 H^{B_2 d^N}.$$

*Proof.* It suffices to prove the theorem when  $a = N + 1$ .

The proof of the Theorem is by induction on  $N$ . The case of  $N = 1$  is standard and proved, for instance, by Bugeaud in [4] Theorem 8.3.

We suppose that the Theorem is true for  $\mathbf{C}^{N-1}$ .

Fix  $\zeta = (\zeta_2, \dots, \zeta_N) \in \mathbf{C}^{N-1}$ . If  $P(z_1, \dots, z_N) \in \mathbf{Z}[z_1, \dots, z_N]$ , we denote by  $P_\zeta(z) \in \mathbf{Z}[\zeta_2, \dots, \zeta_N][z]$  the polynomial  $P(z, \zeta_2, \dots, \zeta_N)$ . If  $\deg(P(z)) = d$  then  $\deg(P_\zeta(z)) \leq d$  (remark that  $P_\zeta[z]$  is a polynomial in  $\mathbf{C}[z]$ ).

Suppose that  $\zeta \in \mathbf{C}^{N-1}$  is an arithmetically generic point (no polynomial with coefficient in  $\mathbf{Z}$  vanish on it). Fix an  $r > 0$ . A complex number  $z \in \Delta_r$  is said to be of type  $S_N(\zeta)$  if there exists a constant  $A = A(z, \zeta)$  such that, for every  $P \in \mathbf{Z}[z_1, \dots, z_N]$  of degree at most  $d$  we have

$$(5.4) \quad \log |P_\zeta(z)| \geq -Ad^{N+1}(\log \|P\| + d^2 \log(d)) + \log \|P_\zeta\|_r$$

(We recall that  $\|P\|_r = \sup_{z \in \Delta_r} \{|P(z)|\}$ ).

We claim that it suffices to prove the following:

**Claim:** For every arithmetically generic  $\zeta \in \mathbf{C}^{N-1}$  and every positive  $r > 0$ , the set of  $z \in \Delta_r$  of type  $S_N(\zeta)$  is full in  $\Delta_r$ .

Indeed, by Fubini Theorem, the claim implies that the subset of points  $(z, \zeta) \in \Delta_r \times \mathbf{C}^{N-1}$  for which there exists a constant  $A$  (depending on  $(z, \zeta)$ ) such that, for every  $P \in \mathbf{Z}[z_1, \dots, z_N]$  inequality 5.4 holds, is full.

Suppose now that  $\zeta \in \mathbf{C}^{N-1}$  is of type  $S_N$ . Let  $P \in \mathbf{Z}[z_1, \dots, z_N]$  of degree  $d$ . Suppose, for the moment that  $z_1 \nmid P$ . Inductive hypothesis implies that

$$(5.5) \quad \log \|P_\zeta\|_r \geq \log |P(0, \zeta)| \geq -B_1 d^N \log \|P\| - B_2 d^2 \log(d) - B_3 d$$

for suitable constants  $B_i$  depending only on  $\zeta$ . If  $z_1 \mid P$ , consider  $P_1 := \frac{P}{z_1^a}$  for a suitable  $a$ . An inequality similar to 5.5 holds for  $P_1$  and maximum modulus principle implies that 5.5 holds.

Consequently, if  $\zeta \in \mathbf{C}^{n-1}$  is of type  $S_N$  and  $z$  is of type  $S_N(\zeta)$ , then  $(z, \zeta)$  is of type  $S_N$ . Thus the claim implies the theorem.

We now prove the claim. Let  $P(z_1, \dots, z_N)$  a non zero polynomial of degree  $d$ . We apply now formula 4.3 with  $P(z) = P_\zeta(z)$  and  $\epsilon = \frac{1}{\|P\|^{B_1 d^{N+1}} d^{B_2 d^2}}$ , where the  $B_i$ 's will be chosen after. By Corollary 4.3 we obtain then that we can find constants  $C_i$  depending only on  $r$  such that, for every polynomial  $P \in \mathbf{Z}[z_1, \dots, z_N]$  of degree  $d$ , we have

$$(5.6) \quad \mu \left( \left\{ z \in \Delta_r \mid |P_\zeta(z)| \leq \frac{\|P_\zeta\|_r}{\|P\|^{B_1 d^{N+1}} d^{B_2 d^2}} \right\} \right) \leq \frac{C_1}{\|P\|^{C_2 d^N} d^{C_3 d}}.$$

Denote by  $A(P)$  the area of the set  $\{z \in \Delta_r \mid |P_\zeta(z)| \leq \frac{\|P_\zeta\|_r}{\|P\|^{B_1 d^{N+1}} d^{B_2 d^2}}\}$ .

In order to conclude, by application of the Borel-Cantelli lemma, we compute

$$\begin{aligned} \sum_{d=1}^{\infty} \sum_{P \in V_d(N)} A(P) &\leq \\ &\leq \sum_{d=0}^{\infty} \sum_{P \in V_d(N)} \frac{B_0}{\|P\|^{B_1 d^N} d^{B_3 d}} \\ &= \sum_{d=1}^{\infty} \sum_{H=1}^{\infty} \sum_{\substack{P \in V_d(N) \\ H \leq \|P\| < H+1}} \frac{B_0}{\|P\|^{B_1 d^N} d^{B_3 d}} \\ &\leq \sum_{d=1}^{\infty} \sum_{H=1}^{\infty} \frac{B_0 H^{B_4 d^N}}{H^{B_1 d^N} d^{B_3 d}} \end{aligned}$$

where the  $B_i$ 's are sufficiently big constants. It is easy to see that the last sum converges as soon as  $B_1 > B_4$  and  $B_3 > 1$ . By Borel-Cantelli Lemma we conclude.  $\square$

## 6. ARITHMETICALLY GENERIC SUB VARIETIES

In the sequel we will study possible generalizations of the Liouville inequalities on transcendental points on varieties. At the moment the inequalities we find are not good enough to be applied to obtain theorems as in the previous section, but we hope that a further analysis will give possible applications.

We fix a smooth projective variety  $X_K$  defined over  $K$  and an arithmetic polarization  $(\mathcal{X}, \overline{\mathcal{L}})$  over it. In this section we will focus our attention on closed sub varieties of  $X_{\sigma_0}$  which are far from being defined over a number field.

**Definition 6.1.** *Let  $Z \subset X_{\sigma_0}(\mathbf{C})$  be a Zariski closed sub variety. We will say that  $Z$  is arithmetically generic (or that  $Z$  is not defined over the algebraic closure of  $K$ ) if,*

for every positive integer  $d$  the natural map

$$H^0(\mathcal{X}, \mathcal{L}^d) \longrightarrow H^0(Z, (\mathcal{L}^d)_{\sigma_0}|_Z)$$

is injective.

It is easy to verify that this definition is independent on the choice of the arithmetic polarization.

The definition means that  $Z$  cannot be contained in a proper sub variety defined over the algebraic closure of  $K$ . In other words, there is at least one generator of the ideal sheaf defining  $Z$  whose coefficients are transcendental. Observe that, with this definition,  $X_{\sigma_0}$  as a sub variety of itself, *is not defined over the algebraic closure of  $K$*  (or it is arithmetically generic). This is a bit unnatural (and not correct) but we prefer to keep the definition like that because this will only affect the following fact: if  $Z$  is a sub variety of  $X_{\sigma_0}$  not defined over the algebraic closure of  $K$ , we may suppose that  $Z$  is  $X_{\sigma_0}$  itself.

We now list some general properties of arithmetic generically sub varieties.

**Lemma 6.2.** *Let  $Z$  be a Zariski closed sub variety of  $X_{\sigma_0}(\mathbf{C})$  which is arithmetically generic. Then the set of irreducible effective Cartier divisors of  $Z$  which are not arithmetically generic, is countable.*

*Proof.* Let  $D$  be an irreducible effective Cartier divisor of  $Z$  which is defined over the algebraic closure of  $K$ . This means that we can find a positive integer  $d$  and a section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  which vanishes along  $D$ . Thus  $\text{div}(s|_Z) = \sum n_i D_i + n_0 D$ . By hypothesis,  $s$  do not vanish identically on  $Z$ . For every integer  $d$  and for every section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  denote by  $DIV(Z, s)$  the set  $\{D_i \text{ div}(Z) / \text{div}(s|_Z) = \sum n_i D_i\}$ . The set of irreducible cartier divisors of  $Z$  which are defined over the algebraic closure of  $K$  is the union of the  $DIV(Z, s)$ 's where  $s$  runs over  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  and  $d$  runs over the positive integers. Since each of the  $H^0(\mathcal{X}, \mathcal{L}^d)$  is countable and each of the  $DIV(Z, s)$  is finite, the conclusion follows.  $\square$

As a consequence of this, we find:

**Proposition 6.3.** *There exist sub varieties of any dimension which are arithmetically generic.*

*Proof.* Let  $D$  be a divisor of  $X_{\sigma_0}$ . The complementary of it is dense in  $X_{\sigma_0}$  for the euclidean topology. Denote it by  $U_D$ . A point  $x \in X_{\sigma_0}$  is not defined over the algebraic closure of  $K$  if it do not belong to  $\text{div}(s)$  where  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  for every positive integer  $d$ . Thus the set of points  $x$  which are not defined over the algebraic closure of  $K$  is the intersection of countably many  $U_D$ 's. Thus, from Baire Lemma we deduce that these points are dense in  $X_{\sigma_0}$ . Every closed variety passing through at least one of these points is not defined over the algebraic closure of  $K$ .  $\square$

We keep in mind the following corollary:

**Corollary 6.4.** *Suppose that  $Z_1 \subset Z_2 \subset X_{\sigma_0}(\mathbf{C})$  are Zariski closed sub varieties. If  $Z_1$  is arithmetically generic then  $Z_2$  is arithmetically generic.*

In particular, if  $Z$  contains an arithmetically generic point, then itself is arithmetically generic.

In this paper we would like to deal with properties which are shared by "almost" all the varieties. We give a definition which clarify the notion of "almost all the varieties":

**Definition 6.5.** *Let  $R$  be a set of sub varieties of  $X_{\sigma_0}$ . We will say that  $R$  is full if, for every fibration  $h : X_K \rightarrow B_K$ , the set of  $b \in B_{\sigma_0}(\mathbf{C})$  such that  $X_b := f^{-1}(b)$  is in  $R$  is either empty or full in  $B_{\sigma_0}$  (for the Lebesgue measure).*

With this definition, as a consequence of the proposition below, the set of arithmetically generic varieties is full:

**Proposition 6.6.** *Let  $f : X_K \rightarrow B_K$  a fibration. Then a point  $b \in B_{\sigma_0}(\mathbf{C})$  is arithmetically generic in  $B_K$  if and only if the fibre  $X_b$  is an arithmetically generic variety in  $X_K$ .*

*Proof.* We first make the following easy observation: suppose that  $Z \subset X_{\sigma_0}$  is a Zariski closed sub variety and  $M_K$  is *any* line bundle over  $X_K$  (thus defined over  $K$ ). Suppose that  $s \in H^0(X_K, M)$  is a global section such that  $s|_Z = 0$ , then  $Z$  is not arithmetically generic (proof:  $\text{div}(s)$  is a component of an ample divisor of  $X_K$ ).

As a consequence of the observation above we have: Suppose that  $X_b$  is arithmetically generic, then  $b$  must be arithmetically generic. Indeed, if not, then there is a divisor  $D$  in  $B$  such that  $D|b = 0$ . Consequently,  $f^*(D)|X_b = 0$  and thus  $X_b$  is not arithmetically generic.

Another consequence of the observation is that if  $X_b$  is finite and  $b \in B_{\sigma_0}(\mathbf{C})$  is arithmetically generic, then every element of  $X_b(\mathbf{C})$  is arithmetically generic. Indeed, if  $D$  is an effective divisor of  $X_K$  passing through  $x$ , then  $f_*(D)$  is an effective divisor of  $B_K$  passing through  $f(x)$ .

Suppose that  $b \in B_{\sigma_0}(\mathbf{C})$  is arithmetically generic. Let  $Y \subset X_K$  be a closed sub variety such that  $Y \cap X_b$  is finite. Then every point of  $Y \cap X_b$  is arithmetically generic, and consequently  $X_b$  is arithmetically generic.  $\square$

We will now show that, for arithmetically generic varieties inequalities of Liouville type as in Theorem 3.2 cannot hold.

**Theorem 6.7.** *We can find a constant  $A$  depending only on the chosen arithmetic polarization for which the following holds: let  $z \in X_{\sigma_0}(\mathbf{C})$  be an arithmetically generic point, then, for every  $d \in \mathbb{N}$  there exists an infinite sequence of sections  $s_n \in H^0(\mathcal{X}, \mathcal{L}^d)$  for which*

$$(6.1) \quad \log \|s_n\|_{\sigma_0}(z) \leq -Ad^{\dim(X_K)}(\log^+ \|s_n\| + d).$$

*Proof.* We denote by  $L$  the line bundle  $\mathcal{L}_{\sigma_0}$ . For every positive integer  $d$ , let  $L_z^d$  be the fibre of  $L^d$  over  $z$ . We put on  $L_z^d$  the unique Haar measure  $\mu$  for which  $\mu(\{v \in$

$L_x^d / \|v\|_{\sigma_0} \leq 1\} = 1$ . For every positive real number  $T$ , we have that  $\mu(\{v \in L_x^d / \|v\|_{\sigma_0} \leq T\}) = T^2$ .

Since  $z$  is an arithmetically generic point, we have an inclusion  $\iota_d : H^0(\mathcal{X}, \mathcal{L}^d) \hookrightarrow L_z^d$ . By Dirichlet box principle and inequality 2.6 we can find positive constants  $C_i$  for which the following holds: as soon as  $T$  is sufficiently big, there are two distinct sections  $s_1$  and  $s_2$  in  $H^0(\mathcal{X}, \mathcal{L}^d)$  such that  $\sup_{\tau \in M_K^\infty} \{\|s_i\|_\tau\} \leq T$  and  $\|s_1 - s_2\|_{\sigma_0}(z) \leq \frac{T^2}{C_1^{d \dim(X_K) + 1} T^{C_2 d \dim(X_K)}}$ . The conclusion follows from the fact that, if  $T > 1$ , then  $\sup_{\tau \in M_K^\infty} \{1, \|s_i\|_\tau\} \leq T$ .  $\square$

Following the path of the proof of the previous theorem we can prove the following

**Theorem 6.8.** *Let  $Y \subset X_{\sigma_0}$  be an arithmetically generic sub variety. Then we can find a constant  $A$  depending on  $Y$  and on the arithmetic polarization for which the following holds: for every  $d \in \mathbb{N}$  there exists an infinite sequence of sections  $s_n \in H^0(\mathcal{X}, \mathcal{L}^d)$  for which*

$$(6.2) \quad \log \|s_n\|_{Y; \sigma_0}(z) \leq -Ad^{\dim(X_K)}(\log^+ \|s_n\| + d).$$

## 7. $S_a$ POINTS ON PROJECTIVE VARIETIES

Let  $X_K$  be a smooth projective variety of dimension  $n$  defined over  $K$ . Fix an arithmetic polarization  $(\mathcal{X}, \mathcal{L})$  of  $X_K$ . In this section we will define a class of arithmetically generic points on  $X_{\sigma_0}(\mathbf{C})$  which verify a kind of weak Liouville inequality. This class is quite natural and in the next sections we will show that it is a set of full Lebesgue measure.

Suppose that  $z \in X_{\sigma_0}(\mathbf{C})$  is a complex point.

**Definition 7.1.** *Let  $a \geq n$  be a real number. We will say that  $z$  is of type  $S_a$  (or that  $z \in S_a(X_K)$ ) if we can find positive constants  $A = A(z, \overline{\mathcal{L}}, a)$  and  $B = B(z, \overline{\mathcal{L}}, a)$  depending on  $z$  and the polarization such that, for every positive integer  $d$  and every non zero global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  we have that*

$$\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -Ad^a(\log^+ \|s\| + Bd).$$

*In this case we will say that  $z$  has transcendental height on  $X_K$  (with respect to the arithmetic polarization and  $a$ ) less or equal then  $A$ . Moreover we will denote by  $S_a(X_K)$  the subset of  $X_{\sigma_0}(\mathbf{C})$  of points of type  $S_a$ .*

Observe that a point of type  $S_a$  is necessarily transcendental, and even in a strong sense: it is not contained in any closed sub variety defined over the algebraic closure of  $K$ .

Remark also that the condition  $a \geq n$  is necessary because of Theorem 6.7.

**Remark 7.2.** *We would like to point out that, when  $z$  is an algebraic point, Liouville inequality 3.1 tells us that the involved constant  $A$  is essentially the height of  $z$ . For this reason, we introduced here the definition of transcendental height.*



We fix  $a \geq n$ .

The set  $S_a(X_K)$  do not depend of the arithmetic polarization. This is proved in the following proposition.

**Proposition 7.3.** *The set  $S_a(X_K)$  is independent on the choice of the arithmetic polarization.*

*Proof.* We begin by fixing an arithmetic polarization on  $X_K$ . Let  $z \in S_a(X_K)$ . For the time being we will say that  $z$  is of type  $S_a$  with respect to the fixed arithmetic polarization.

First remark that, if  $\mathcal{L}$  is the involved ample line bundle on the involved model  $\mathcal{X}$  and  $d_0$  is a positive integer, then, for every positive integer  $d$  and every global section  $s \in H^0(\mathcal{X}, \mathcal{L}^{d_0d})$  we have that

$$\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -(Ad_0)^a d^a (\log^+ \|s\| + (Bd_0)d).$$

Conversely, suppose that we can find constants  $A_1$  and  $B_1$  such that, for every positive integer  $d$  and every  $s \in H^0(\mathcal{X}, \mathcal{L}^{d_0d})$  we have that  $\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -A_1 d^a (\log^+ \|s\| + B_1 d)$ . If  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ , then,  $s^{d_0} \in H^0(\mathcal{X}, \mathcal{L}^{d_0d})$ . Since  $\log \|s^{d_0}\|_{\sigma_0}(z) = d_0 \log \|s\|_{\sigma_0}(z)$  and  $\log^+ \|s^{d_0}\| \leq d_0 \log^+ \|s\|$ , we have that

$$\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -(Ad_0)^a d^a (\log^+ \|s\| + Bd).$$

Hence, changing  $\mathcal{L}$  with  $\mathcal{L}^{d_0}$  do not affect the set  $S_a$ .

*Independence on the the change of metrics:* For every  $\sigma \in M_K^\infty$ , denote by  $\|\cdot\|_\sigma$  the metric on the involved very ample line bundle  $L_\sigma$ . Suppose that we change the metric on  $L_\tau$  with  $\tau \neq \sigma_0$ . Denote by  $\|\cdot\|_\tau^1$  the new metric. Since  $X_\tau$  is compact, we can find a constant  $C$  such that, for every positive integer  $d$  and every global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ , we have  $\log^+ \|s\|^1 \leq \log^+ \|s\| + Cd$ . Hence changing the metric at a place different from  $\sigma_0$  do not affect  $S_n(X_K)$ . Suppose now that we change the metric on  $\mathcal{L}_{\sigma_0}$ . Denote again  $\|\cdot\|_{\sigma_0}^1$  the new metric. We can find constants  $C_1$  and  $C_2$  such that, for every positive integer  $d$  and every global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ , we have  $\log^+ \|s\|^1 \leq \log^+ \|s\| + C_2 d$ , and  $\|s_{\sigma_0}\|_{\sigma_0}(z) \geq \|s_{\sigma_0}\|_{\sigma_0}^1(z) + C_2 d$ . Thus the change of the metric at the place  $\sigma_0$  do not affect  $S_n(X_K)$ .

*Independence on the involved ample line bundle  $\mathcal{L}$ .* Suppose that  $\mathcal{M}$  is a relatively ample line bundle on the model  $\mathcal{X}$ . Replacing  $\mathcal{L}$  and  $\mathcal{M}$  by a positive power if necessary, we may suppose that there exists an effective divisor  $D$  on  $\mathcal{X}$  such that  $\mathcal{L} = \mathcal{M}(D)$ . We may choose metrics  $\|\cdot\|^\mathcal{L}$ ,  $\|\cdot\|^\mathcal{M}$  and  $\|\cdot\|^D$  on  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{O}_\mathcal{X}(D)$  respectively in such a way that this is an isometry (these changes will not affect the points  $S_n(X_K)$  with respect to both arithmetic polarizations). Denote by  $s_D \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}(D))$  a section such that  $\text{div}(s_D) = D$ . Observe that  $s_D(z) \neq 0$ .

Suppose that  $s \in H^0(\mathcal{X}, \mathcal{M}^d)$  is a non zero section. Then  $\tilde{s} := s \otimes s_D^d \in H^0(\mathcal{X}; \mathcal{L}^d)$ . If we denote by  $B_1$  the real number  $\log \|s_{D, \sigma_0}\|_{\sigma_0}(z)$ , we have that  $\log \|\tilde{s}_{\sigma_0}\|_{\sigma_0}^\mathcal{L}(z) = \log \|s_{\sigma_0}\|_{\sigma_0}^\mathcal{M}(z) + dB_1$ . Moreover we can find a constant  $B_2$  such that  $\log^+ \|\tilde{s}\| \leq \log^+ \|s\| + dB_2$ . Hence if  $z$  is of type  $S_a$  with respect to  $\mathcal{L}$  then  $z$  is of type  $S_a$

with respect to  $\mathcal{M}$ . Exchanging the roles of  $\mathcal{L}$  and  $\mathcal{M}$  the independence on  $\mathcal{L}$  follows.

*Independence on the involved model  $\mathcal{X}$ .* Suppose that we have two models  $\mathcal{X}_1$  and  $\mathcal{X}_2$  of  $X_K$ . We may suppose that  $\mathcal{X}_1$  is a blow up of  $\mathcal{X}_2$  with exceptional divisor  $E$ . Thus we have a birational morphism  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  which may supposed to be the identity on the generic fiber. We may suppose that  $\mathcal{L}_2$  is an ample line bundle on  $\mathcal{X}_2$  such that  $\mathcal{L}_1 := f^*(\mathcal{L}_2)(-E)$  is ample on  $\mathcal{X}_1$ . Let  $z \in X_{\sigma_0}(\mathbf{C})$ . The fact that if  $z$  is of type  $S_a$  with respect to  $\mathcal{L}_1$  then it is of type  $S_a$  with respect to  $\mathcal{L}_2$  follows the same path of the proof above (independence on the line bundle) thus it is left to the reader.

Observe that  $E$  is a vertical divisor, thus there is an integer  $N$  with the following property: Let  $dE$  be the  $d$ -th infinitesimal neighborhood of  $E$ ; suppose that  $\mathcal{M}$  is a line bundle on  $\mathcal{X}_1$  and  $g \in H^0(dE, \mathcal{M}|_{dE})$  then  $N^d \cdot g = 0$ . This is proved by induction on  $d$  by using the exact sequence

$$0 \longrightarrow \mathcal{M}(-dE)|_E \longrightarrow \mathcal{M}|_{dE} \longrightarrow \mathcal{M}|_{(d-1)E} \longrightarrow 0.$$

For every positive integer  $d$ , we have an exact sequence on  $\mathcal{X}_1$

$$0 \longrightarrow f^*(\mathcal{L}_2^d) \longrightarrow \mathcal{L}_1^d \longrightarrow \mathcal{L}_1^d|_{dE} \longrightarrow 0.$$

Hence, if  $s \in H^0(\mathcal{X}_1, \mathcal{L}_1^d)$  then  $N^d \cdot s \in H^0(\mathcal{X}_2, \mathcal{L}_2^d)$ . Consequently, if  $z$  is of type  $S_a$  with respect to  $\mathcal{L}_2$  then it is of type  $S_a$  with respect to  $\mathcal{L}_1$ .  $\square$

As a consequence of proposition above, it is correct to do not mention the arithmetic polarization when we consider the set  $S_a(X_K)$ .

**Remark 7.4.** *It is interesting to remark that the proof of the independence of  $S_a(X_K)$  on the polarization is very similar to the proof, in Arakelov geometry, of the independence of the height definition on the choice of models and metrics.*

## 8. ARITHMETICALLY GENERIC CURVES

In this section we would like to study properties analogues to the property  $S_a$  on points which belong to the same arithmetically generic curve (Zariski sub variety of dimension one).

At the moment we are not able to attack the case when  $n \leq a < n+1$ . Thus, from now on we are going to suppose that

$$(8.1) \quad a \geq n+1.$$

Again we fix a smooth projective variety  $X_K$  defined over  $K$  and an arithmetic polarization  $(\mathcal{X}, \overline{\mathcal{L}})$  over it. We also fix an arithmetically generic closed curve  $Y \subset X_{\sigma_0}(\mathbf{C})$ .

Again, if  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  is a non zero section, we will denote

$$\log \|s\|_Y := \sup\{\log \|s\|_{\sigma_0}(z) / z \in Y\}.$$

Remark that  $\log \|s\|_Y$  is a real number, because, since  $Y$  is arithmetically generic, none non zero section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  will vanish identically on  $Y$ .

In analogy with the definition 7.1 we give:

**Definition 8.1.** *We will say that  $z \in Y$  is of type  $S_a^Y$  (or that  $z \in S_a(Y)$ ) if we can find positive constants  $A = A(z, \overline{\mathcal{L}}, Y, a)$  and  $B = B(z, \overline{\mathcal{L}}, Y, a)$  depending on  $z$ , the polarization and  $Y$  and  $a$  such that, for every positive integer  $d$  and every non zero global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  we have that*

$$\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -Ad^a(\log^+ \|s\| + Bd) + \log \|s\|_Y.$$

Moreover we will denote by  $S_a(Y)$  the subset of  $Y$  of points of type  $S_a^Y$ .

Remark that the main difference between points of  $S_n(X_K)$  and points of  $S_n(Y)$  is in the last term on the right of the inequality. A priori  $\log \|s\|_Y$  may be much smaller than  $\log^+ \|s\|$  (besides the fact that we are not considering the  $\log^+(\cdot)$ ).

The same proof (mutatis mutandis) of Proposition 7.3 gives:

**Proposition 8.2.** *The set  $S_a(Y)$  does not depend on chosen the arithmetic polarization.*

The main theorem of this section is the following:

**Theorem 8.3.** *With the notations as above, then the set  $S_a(Y)$  is full in  $Y$ .*

An important corollary of this theorem is

**Corollary 8.4.** *If  $Y$  is an arithmetically generic curve and  $Y \cap S_a(X_K) \neq \emptyset$  then  $S_a(X_K) \cap Y$  is full in  $Y$ .*

Thus, we find the following important principle:

*Either an arithmetic generic curve is contained in the complementary of  $S_a(X_K)$ , or its intersection of it with  $S_a(X_K)$  is almost all the curve itself.*

In order to prove Theorem 8.3, we may suppose that  $a = n + 1$ .

The proof of Theorem 8.3 requires three lemmas of different nature.

The first Lemma is Lemma 4.1.

The second Lemma is an easy consequence of the following Theorem due to Sadullaev (we present it here in the form we need):

**Theorem 8.5.** *Let  $Z \subset \mathbf{C}^n$  be an analytic sub variety of dimension one. Let  $U_1 \subset U_2$  be two relatively compact open sets of  $Z$ . Then the followings are equivalent:*

- a)  $Z$  is an open set of an affine curve (that means that  $Z$  is algebraic);*
- b) There exists a constant  $C$  (depending only on the  $U_i$ 's and  $n$ ) with the following property: For ever polynomial  $P(z) = P(z_1, \dots, z_n) \in \mathbf{C}[z_1; \dots, z_n]$  we have that*

$$\sup_{z \in U_2} \{|P(z)|\} \leq C^{\deg(P)} \sup_{z \in U_1} \{|P(z)|\}.$$

For a proof, cf. [14] Theorem 2.2.

From theorem 8.5 we easily deduce the following:

**Lemma 8.6.** *Let  $Y \subset X_{\sigma_0}$  as before. Let  $U \subset Y$  be a (non compact) simply connected open set whose complementary has non empty interior. Fix a biholomorphic isomorphism  $\varphi_U : \Delta_1 \rightarrow U$ . Fix two positive integers  $0 < r_0 < r_1 < 1$ . Then we can find a constant  $C_1$  such that, for every positive integer  $d$  and every  $s \in H^0(X_{\sigma_0}, \mathcal{L}_{\sigma_0}^d)$  we have that*

$$\varphi_U^*(s) \in B_{C_1}(r_0, r_1, \mathcal{L}_{\sigma_0}^d).$$

The definition of  $B_{C_1}(r_0, r_1, \mathcal{L}_{\sigma_0}^d)$  is given in 4.1.

*Proof.* Taking a power of  $\mathcal{L}$  if necessary, we may suppose that  $\mathcal{L}$  is very ample. By formula 4.2 we may suppose that  $X_K$  is the projective space  $\mathbf{P}^N$ ,  $\mathcal{L} = \mathcal{O}(1)$  and that there is an hyperplane  $H$  which do not intersect  $U$ . Let  $V := \mathbf{P}^N \setminus H$  and  $Z = Y \cap V$ . We are in position to apply Theorem 8.5 and conclude by choosing the suitable constants needed to compare the involved metrics.  $\square$

The last lemma is of topological nature. We would like to thank T. Delzant who provided the main idea of it.

**Lemma 8.7.** *Let  $Z$  be a compact Riemann surface. Then we can find a finite set of coverings  $\mathcal{U}_j = \{U_{ij}\}$ ,  $j = 1, \dots, r$  with the following properties:*

- a) *The set of open set  $\{U_{ij}\}$  is finite.*
- b) *Each  $U_{ij}$  is non compact and simply connected. Fix an biholomorphic isomorphism  $\varphi_{ij} : \Delta_1 \rightarrow U_{ij}$ .*
- c) *Fix a positive real number  $0 < r_0 < 1$ . Denote by  $U_{ij}^{r_0}$  the open set  $\varphi_{ij}^{-1}(\Delta_{r_0})$ . Then, for each  $j$ ,  $Z = \cup_i U_{ij}^{r_0}$  (this means that, for  $j$  fixed, the open sets  $U_{ij}^{r_0}$  still cover  $Z$ ).*
- d) *For every  $z \in Z$  there is a  $j_z$  such that  $z \in \cap_{ij_z} U_{ij_z}$ .*

*Proof.* Fix a finite covering  $\mathcal{U}_0 = \{U_i\}$  of  $Z$  such that:

- Each  $U_i$  is simply connected. Fix an biholomorphic isomorphism  $\varphi_i : U_i \rightarrow \Delta_1$  and denote by  $U_i^0$  the open set  $\varphi_i^{-1}(\Delta_{r_0})$ .
- $Z = \cup_i U_i^0$  and  $\cap_i U_i^0 \neq \emptyset$ .

Denote by  $V := \cap_i U_i^0$ . Fix a point  $z_0 \in V$ .

For each  $z \in Z$  we fix a diffeomorphism  $f_z : Z \rightarrow Z$  such that  $f(z) = z_0$ . Hence  $Z = \cup_{z \in Z} f_z^{-1}(V)$ . Since  $Z$  is compact, we can find a finite set  $z_1, \dots, z_n$  such that  $Z = \cup_{h=1}^n f_{z_h}^{-1}(V)$ . For each  $h = 1, \dots, n$  denote by  $\mathcal{U}_h$  the covering  $\{f_{z_h}^{-1}(U_i)\}$ .

It is easy to verify that the collection of coverings  $\mathcal{U}_h$ ,  $h = 1, \dots, n$  has the searched properties.  $\square$

We can now prove Theorem 8.3.

*Proof. (of Theorem 8.3)* We fix a finite set of coverings  $\mathcal{U}_j$  of  $Y$  which verify Lemma 8.7.

For every positive integer  $d$ , positive real constants  $B_i$  and non zero section  $s \in H^0(\mathcal{X}, \mathcal{L}_{\sigma_0}^d)$  denote by  $V_Y(s, B_1, B_2)$  the area of the set  $\{z \in Y / \|s\|_{\sigma_0} \leq \frac{\|s\|_Y}{\|s\|_{+}^{B_1 d^{n+1}} B_2^{d^{n+2}}}\}$ .

We now claim the following:

For every sufficiently positive constant  $C_0$  we can find constants  $C_i > 1$  depending only on  $Y$ , the arithmetic polarization and the constants  $B_i$  for which the following holds:

For every positive integer  $d$  and non zero section  $s \in H^0(\mathcal{X}, \mathcal{L}_{\sigma_0}^d)$ , we have

$$(8.2) \quad V_Y(s, B_1, B_2) \leq \frac{C_0}{\|s\|_+^{C_1 d^n} C_2^{d^{n+1}}}.$$

*Proof of the claim:* Fix  $s \in H^0(Y, \mathcal{L}_{\sigma_0}^d)$ . Let  $z_0 \in Y$  such that  $\sup_{z \in Y} \{\|s\|_{\sigma_0}\} = \|s\|_{\sigma_0}(z_0)$ . we can find one of the coverings  $\mathcal{U}_j$  with the property that  $z_0$  belongs to all the open sets of the covering. We may apply Lemma 8.6 and Lemma 4.1 to each of the open sets of the involved covering with  $\epsilon = 1/(\|s\|_+^{B_1 d^{n+1}} B_2^{d^{n+2}})$  and the claim follows.

An important issue which follows from the proof of the claim (and that will be used in the following) is that  $\lim_{B_i \rightarrow +\infty} C_j = +\infty$ . By this we mean that, up to increase the  $B_i$  if necessary, we may suppose that the  $C_j$  are as big as we want.

We are now in position to conclude the proof of the Theorem. The conclusion will be a direct application of Borel–Cantelli Lemma 2.3.

Fix constants  $B_i$  sufficiently big (how big will be fixed at the end of the proof). By the claim above we have that (for a suitable constant  $C_3$ ):

$$\begin{aligned} & \sum_{d=1}^{\infty} \sum_{s \in H^0(\mathcal{X}; \mathcal{L}^d)} V(s, B_1, B_2) \\ & \leq \sum_{d=0}^{\infty} \sum_{s \in H^0(\mathcal{X}; \mathcal{L}^d)} \frac{C_0}{\|s\|_+^{C_1 d^n} C_2^{d^{n+1}}} \quad \text{by formula 8.2} \\ & = \sum_{d=1}^{\infty} \sum_{N=1}^{\infty} \sum_{\substack{s \in H^0(\mathcal{X}; \mathcal{L}^d) \\ N \leq \|s\|_+ < N+1}} \frac{C_0}{\|s\|_+^{C_1 d^n} C_2^{d^{n+1}}} \\ & \leq \sum_{d=1}^{\infty} \sum_{N=1}^{\infty} \frac{C_0 C_3^{d^{n+1}} N^{d^n}}{N^{C_1 d^n} C_2^{d^{n+1}}} \quad \text{by formula 2.6} \end{aligned}$$

and the last series converges as soon as  $C_2$  is sufficiently big and  $C_1 > 1$ . By the Borel Cantelli lemma 2.3 the conclusion follows.  $\square$

## 9. $S_a$ POINTS ARE FULL IF $a \geq n + 1$

We suppose that we are in the same hypotheses of the previous section.

As Theorem 6.8 shows, the term  $\log \|s\|_Y$  is, in general, quite difficult to control. It is possible that it is much smaller than the term  $\log^+ \|s\|$ . Never the less we will now show that the set  $S_a(X_K)$  is a full set in  $X_{\sigma_0}(\mathbf{C})$ .

**Theorem 9.1.** *If  $a \geq n + 1$ , the set  $S_a(X_K)$  is full in  $X_{\sigma_0}(\mathbf{C})$ .*

*Proof.* The proof is by induction on the dimension of the variety. As before we may suppose that  $n = \dim(X_K) \geq 2$ , that  $a = n + 1$ , and that, for every generically smooth divisor  $D$  of  $X_K$ , the set  $S_n(D_\sigma)$  is full in  $D_\sigma(\mathbf{C})$ .

Fix a section  $s_0 \in H^0(\mathcal{X}, \mathcal{L})$  and denote by  $\mathcal{D} \subset \mathcal{X}$  the divisor  $\text{div}(s_0)$ . By Bertini theorem we may suppose that its generic fiber  $D_K$  is smooth. Fix a point  $z \in S_n(D_{\sigma_0}(\mathbf{C}))$ . The set of these points is full in  $D_{\sigma_0}(\mathbf{C})$  by inductive hypothesis.

Let  $Y \subset X_{\sigma_0}(\mathbf{C})$  be a closed arithmetically generic curve passing through  $z$ .

The proof will be a consequence of this lemma, the proof of which will be postponed to the end of the proof.

**Lemma 9.2.** *There is a constant  $A$ , depending only on  $Y$  and  $z$  for which the following holds: For every positive integer  $d$  and non vanishing section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  we have that*

$$\log \|s\|_Y \geq -Ad^n(\log^+ \|s\| + d).$$

Let's show how the lemma implies the theorem. Theorem 8.3 and Lemma 9.2 imply that, if  $z \in Y$  and  $Y$  is arithmetically generic, then  $S_n(X_K) \cap Y$  is full in  $Y$ . Choose a fibration by curves  $h : X_K \rightarrow B$  for which the induced morphism  $h_{D_K} : D_K \rightarrow B$  is generically finite. The image  $h_{D_K}(S_n(D_{\sigma_0}(\mathbf{C})))$  will be full in  $B$ . Consequently Fubini Theorem implies the conclusion of the Theorem.  $\square$

*Proof. (of Lemma 9.2)* Let  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  a non vanishing section. If  $s(z) \neq 0$  then, since  $z \in S_n(D_{\sigma_0}(\mathbf{C}))$  and the restriction of  $s$  to  $D$  is non zero, we have that

$$\log \|s\|(z) \geq -Ad^n(\log^+ \|s\| + d)$$

for a suitable constant  $A$  independent on  $s$  and  $d$ . Thus, in this case, the conclusion of the Lemma holds for  $s$  (because  $z \in Y$ ).

Since  $Y$  is arithmetically generic, the restriction of  $D_{\sigma_0}$  to  $Y$  is not identically zero. Thus, the restriction of  $\text{div}(s_0)$  to  $Y$  is an effective divisor vanishing on  $z$ . Denote by  $b$  the multiplicity at  $z$  of  $\text{div}(s_0)|_Y$  and by  $\alpha$  the real number  $\log \|J^b(s_0)\|$ , where  $J^b(s_0)$  is the  $b$ -th jet at  $z$  of the section  $s_0|_Y$ .

Suppose now that  $s(z) = 0$ . This implies that we can find a constant  $\alpha \leq d$  such that  $\text{div}(s) = \alpha D + D_1$  with  $D_1$  not vanishing on  $z$ . Indeed, since  $z$  is arithmetically generic on  $D$ , every non zero global section of  $\mathcal{L}^d|_D$  will not vanish on  $z$ . Consequently the order of vanishing at  $z$  of the restriction of  $s$  to  $Y$  is  $\alpha b$ . Denote by  $J^{\alpha b}(s)$  the  $\alpha b$ -th jet of  $s|_Y$  in  $z$ .

Denote by  $s_1 \in H^0(\mathcal{X}, \mathcal{L}^{d-\alpha})$  the global section  $s/s_0$ . Since  $s_1$  do not vanish at  $z$ , as before, we have that

$$(9.1) \quad \log \|s_1\|_{\sigma_0}(z) \geq -Ad^n(\log^+ \|s_1\| + d)$$

for a suitable constant  $A$  independent on  $s_1$ . A local computation give the existence of a constant  $C_1$  such that

$$(9.2) \quad \log \|J^{\alpha b}(s)\|(z) \geq \log \|J^\alpha(s_0)\|(z) + \log \|s_1\|_{\sigma_0}(z) + C_1 d.$$

Thus, from 9.1 and 9.2 we conclude that, we can find a constant  $A_2$  independent on the section  $s$ , such that

$$\log \|J^{\alpha b}(s)\|(z) \geq -A_2 d^n(\log^+ \|s_1\| + d).$$

which, together with 2.5, gives the existence of a constant  $A_3$  (independent on  $s$ ) such that

$$(9.3) \quad \log \|J^{\alpha b}(s)\|(z) \geq -A_3 d^n(\log^+ \|s\| + d).$$

Let  $\Delta_1 \subset Y$  be a holomorphic disk centered in  $z$  and with coordinate  $\zeta$ . By Nevanlinna first main theorem applied to the inclusion  $\Delta_1 \hookrightarrow X_{\sigma_0}(\mathbf{C})$  we obtain that there exists a constant  $C$  such that

$$d \cdot C + \int_{|\zeta|=1} \log \|s\|_{\sigma_0} d\theta \geq \log \|J^{\alpha b}(s)\|(z).$$

Thus there exists a point  $\zeta_0 \in Y$  such that

$$\log \|s\|(\zeta_0) \geq -A_3 d^n(\log^+ \|s\| + d).$$

□

We would like to propose the following conjecture which, if true, may imply further interesting speculations:

**Conjecture 9.3.** *Let  $X$  be a smooth projective variety defined over a number field of dimension  $n$  defined over a number field. Let  $a \geq n$  be a real number. Then the set of points of type  $S_a(X)$  is full in  $X(\mathbf{C})$ .*

Even if not stated in this language, the conjecture above is proved, for instance in [4] when  $X = \mathbf{P}^1$ .

## 10. SOME APPLICATIONS TO RATIONAL POINTS ON ANALYTIC DISKS

Even if this paper is more about transcendental points on algebraic varieties, we will see how this theory can be an interesting tool to study rational points.

As before, we suppose that  $X$  is a smooth projective variety of dimension  $n > 1$  defined over a number field  $K$ . We fix an arithmetic polarization  $(\mathcal{X}, \mathcal{L})$  of  $X$ .

Let  $\sigma$  be an infinite place of  $K$  and  $f : \Delta_1 \rightarrow X_\sigma$  be an analytic map ( $\Delta_1$  being the unit disk in  $\mathbf{C}$ ). We are interested on studying the set  $f^{-1}(X(K))$ .

For every positive real numbers  $T$ , and  $r < 1$  we introduce the set

$$(10.1) \quad S_r(f, T) := \{z \in \Delta / |z| < r ; f(z) \in X(K) \text{ and } h_{\mathcal{L}}(f(z)) \leq T\}$$

and we denote by  $C_r(f, T)$  its cardinality.

In this section we are interested in estimating  $C_r(f, T)$  when  $T$  goes to infinity.

In the classical paper [2], authors show that, for every positive  $\epsilon$  we have an estimate of the form  $C_r(f, T) \ll \exp(\epsilon T)$  and examples by Pila [13] and Surroca [15] (independently) show that, in general, one cannot hope better than this.

If some conditions are given on  $f$  then the bound by Bombieri and Pila can be drastically improved. A huge literature on this topic is available, cf. for instance [3], [6], [9], [12].

In this section we will show how points of type  $S_a$  may be used to control the growth of rational points in the image of the analytic map  $f$ . We think that this circle of ideas may be expanded and we hope that we will improve this in a future paper.

Problems similar to this have been studied by many authors: for instance Lang treat the case of maps analytic maps of  $\mathbf{C}$  in his classical book [11]. Two papers which inspire this one are the already quoted [12] and [3].

The main theorem of this section is

**Theorem 10.1.** *Suppose that there is  $z_0 \in \Delta$  such that  $f(z_0) \in S_a(X)$ . Then, for every  $\epsilon > 0$  and  $\gamma > \frac{1}{n}$  there exists a constant  $A = A(\mathcal{X}, \mathcal{L}, r, f, \epsilon, \alpha, \gamma)$  such that, if  $T \geq A$ , we have*

$$(10.2) \quad C_f(r, T) \leq \epsilon T^{1+\gamma(a+1)}.$$

Observe that the hypothesis of the theorem imply that the image of  $f$  is Zariski dense (over the algebraic closure of  $K$ ).

The main tool of the proof is the following Lemma, of independent interest, the proof of which is inspired by the proof of Proposition 2 of [12].

**Lemma 10.2.** *Let  $f : \Delta \rightarrow X_\sigma$  be an analytic map whose image is Zariski dense. Fix  $1 > \epsilon > 0$  and  $\gamma > \frac{n}{n-1}$ . With the notations as above, there is a constant  $A_0 = A_0(\mathcal{X}, \mathcal{L}, f, r, \epsilon, \gamma)$  for which the following holds: for every  $T \geq A_0$  there exists a non zero global section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  such that:*

- $d \leq \epsilon T^{\frac{\gamma}{n}}$ ;
- $\log \|s\| \leq \epsilon T^{1+\frac{\gamma}{n}}$ ;
- For every  $z \in S_r(f, T)$  we have  $s(f(z)) = 0$ .

*Proof.* Before we start the proof, we recall the following standard fact: if we denote by  $h^0(X, L^d)$  the rank of  $H^0(\mathcal{X}, \mathcal{L}^d)$ , then, can find a positive constant  $B_1$  such that, for every  $\epsilon_1 > 0$  and  $d$  sufficiently big, we have

$$(10.3) \quad B_1(1 - \epsilon_1)d^n \leq h^0(X, L^d) \leq B_1(1 + \epsilon_1)d^n.$$

We fix such a  $\epsilon_1 < \frac{1}{9}$ . We also fix  $\epsilon_2 < (\epsilon/10)^n$ .



We also recall that, by Zhang's theorem [17], we may suppose that  $H^0(\mathcal{X}, \mathcal{L}^d)$  is generated by sections of norm less or equal then one.

We may suppose that,  $A_0$  is so big that for  $T \geq A_0$  we have that  $\epsilon_2(1 - \epsilon_1)B_1T^\gamma > 1$  and  $\epsilon_2^{1/n}(4^{1/n} - 3^{1/n})T^{\gamma/n} > 1$ .

Let  $A(T)$  be a positive integer such that  $\epsilon(1 - \epsilon_1)B_1T^\gamma \leq A(T) \leq 2\epsilon_2(1 - \epsilon_1)B_1T^\gamma$ . Choose a subset  $H(T)$  of  $S_r(f, T)$  of cardinality  $A(T)$ .

For every positive integer  $d$ , denote by  $V(T, d)$  the hermitian  $O_K$  module  $\oplus_{z \in H(T)} \mathcal{L}^d|_{f(z)}$ . The rank of  $V(T)$  is  $A(T, \epsilon)$  and  $\mu_{\max}(V(T, d)) \leq dT$ .

We have a natural restriction map

$$(10.4) \quad \delta_T : H^0(\mathcal{X}, \mathcal{L}^d) \longrightarrow V(T, d).$$

By Gromov theorem 2.4, if we put on  $H^0(\mathcal{X}, \mathcal{L}^d)$  the  $L_2$  hermitian structure and on  $V(T, d)$  the direct sum hermitian structure, the norm of  $\delta$  is bounded by  $C_0d$  for a suitable constant  $C_0$ .

We choose  $d$  such that  $3\epsilon_2T^\gamma \leq d^n \leq 4\epsilon_2T^\gamma$ .

Denoty by  $K(T)$  the kernel of  $\delta_T$  and by  $k(T)$  its rank. With our choices we have that

$$(10.5) \quad \frac{h^0(\mathcal{X}; \mathcal{L}^d)}{k(T)} \leq \frac{B_1(1 + \epsilon_1)4\epsilon_2T^\gamma}{B_1(1 - \epsilon_1)3\epsilon_2T^\gamma - 2\epsilon_2(1 - \epsilon_1)B_1T^\gamma} = \frac{4(1 + \epsilon_1)}{(1 - \epsilon_1)} < 5.$$

We may apply Siegel Lemma 2.2 and we find that, for  $T$  sufficiently big, there exists a non vanishing section  $s \in H^0(\mathcal{X}, \mathcal{L})$  with  $d \leq (4\epsilon_2)^{1/n}T^{\gamma/n}$  such that:

- $\log \|s\| \leq 10\epsilon_2^{1/n}T^{1+\frac{\gamma}{n}}$ ;
- for every  $z \in H(T)$  we have  $s(f(z)) = 0$ .

We will now show that, under the hypotheses above, the section  $s$  we just constructed vanishes on every element of  $S_r(f, T)$ .

Let  $z_0$  be an element of  $S_r(f, T)$  which is not in  $H(T)$ .

By taking an automorphism of  $\Delta_1$  we may suppose that  $z_0 = 0$ . The reader will check, that by the compactness of  $\Delta_r$ , the constants which will appear in the estimates below, may be chosen independently on  $z_0$  and depending only on  $r$ .

Suppose that  $s(f(0)) \neq 0$ .

By the standard Liouville inequality, cf. the proof of 3.1, we have that

$$(10.6) \quad \log \|s\|(f(0)) \geq -dT - ([K : \mathbf{Q}] - 1) \log \|s\| \geq -A_2T^{1+\frac{\gamma}{n}}$$

for a suitable constant  $A_2$  independent on  $z_0$ .

Chose  $r_1 > r$  such that, for every  $z \in \Delta_r$  there is an automorphism  $\varphi_z$  of  $\Delta$  which sends  $z$  in 0 and such that  $\varphi(\Delta_r) \subset \Delta_{r_1}$ . We apply Nevanlinna first main theorem:

$$(10.7) \quad dT_f(r_1) + \int_{|z|=r_1} \log \|s\|(r_1 e^{i\theta}) d\theta \geq \sum_{z \in H(T)} \log \frac{r_1}{|z|} + \log \|s\|(f(0))$$

Which, together with the Liouville inequality above and the properties of  $s$  gives the existence of constants  $A_i$  independent on  $z_0$  and on  $T$ , such that

$$(10.8) \quad A_4 T^{1+\frac{\gamma}{n}} \geq A_5 T^\gamma$$

But this cannot hold because of our choice of  $\gamma$ , as soon as  $T$  is sufficiently big. Consequently  $s$  should vanish on  $f(z_0)$  and the conclusion of the Lemma follows.  $\square$

We can now prove Theorem 10.1:

*Proof.* As in the proof of 10.2, we may suppose that  $z_0 = 0$ .

Consider the section  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  we constructed in Lemma 10.2.

Since  $f(0) \in S_n(X)$ , we can choose  $\epsilon$  in such a way that

$$(10.9) \quad \log \|s\|(f(0)) \geq -Ad^a(\log \|s\| + d) \geq -\epsilon T^{a\gamma+1+\gamma}.$$

By Nevanlinna First Main Theorem applied to  $s$ , we thus can find constants  $C_i$  independent on  $T$  such that

$$(10.10) \quad dC_0 + \int_{|z|=r_1} \log \|s\|(r_1 e^{i\theta}) d\theta \geq C_r(f, T)C_1 - \epsilon T^{\gamma(a+1)+1}$$

The conclusion follows from our choice of  $d$ , the bound on  $\log \|s\|$  and a suitable choice of  $\epsilon$ .  $\square$

Theorem 10.1 tells us that, as soon as the image of an analytic map from a disk to  $X$  contains an arithmetically generic point with some good metric properties, then it contains "few" rational points.

We now show that a minor modification of the proof tells us that a similar result is obtained if the image intersect an effective ample divisor in a "good" point:

**Theorem 10.3.** *Let  $f : \Delta_1 \rightarrow X(\mathbf{C})$  as before. Let  $a \geq n - 1$  be a real number. Let  $s_0 \in H^0(\mathcal{L}, \mathcal{L})$  be an irreducible smooth divisor. Suppose that there is  $p \in f(\Delta_r) \cap \text{div}(s_0)$  which is of type  $S_a(\text{div}(s_0))$ . Then, for every  $\epsilon > 0$  and  $\gamma \geq \frac{1}{n}$  there exists a constant  $A = A(\mathcal{X}, \mathcal{L}, r, f, \epsilon, \alpha)$  such that, if  $T \geq A$  then we have*

$$(10.11) \quad C_f(r, T) \leq \epsilon T^{\gamma(a+1)+1}.$$

*Proof.* Since the intersection of  $f(\Delta_r)$  and  $\text{div}(s_0)$  is finite, we can consider just points which are not in  $\text{div}(s_0)$ . Let  $s \in H^0(\mathcal{X}, \mathcal{L}^d)$  be the section constructed via 10.2.

Write  $\text{div}(s) = \alpha \text{div}(s_0) + \text{div}(s_1)$ . By construction, the restriction of  $s_1$  to  $\text{div}(s_0)$  do not vanish identically. By inequality 2.5 we can find positive constant  $C_1$  and  $\epsilon_1$  such that:

$$(10.12) \quad \epsilon T^{\frac{n}{n-1}+\epsilon_1} \geq \log \|s\| \geq \alpha \log \|s_0\| + (d - \alpha)C_1 + \log \|s_1\|.$$

Thus, by our choice of  $d$ , increasing  $T$  and modifying  $\epsilon$  if necessary, we have

$$(10.13) \quad \epsilon T^{\frac{n}{n-1}+\epsilon_1} \geq \log \|s_1\|.$$

Since  $s_1$  do not vanish on  $\operatorname{div}(s_0)$  it do not vanish on  $p$  (because, in particular  $p$  is arithmetically generic in  $\operatorname{div}(s_0)$ ). Consequently, since  $p$  is of type  $S^a(\operatorname{div}(s_0))$ ,

$$(10.14) \quad \log \|s_1\|(p) \geq Ad^a(\log \|s_1\| + d).$$

The conclusion follows as in the proof of Theorem 10.1.  $\square$

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